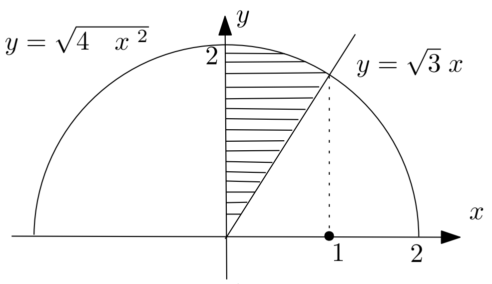
MTS 225 İntegral Hesap (2018-19) Dönem Sonu Sınavı Çözümler

1.



$0 \le x \le 1,\ \sqrt{3}\,x \le y \le \sqrt{4-x^2}$ eşitsizlikleri, yanda görüldüğü gibi, daire dilimini şeklindeki (B bölgesini) belirtir. Bunun sonucu olarak

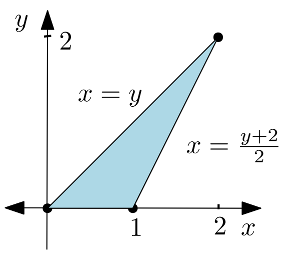
$$\int_0^1 \left(\int_{\sqrt{3}\,x}^{\sqrt{4-x^2}} \frac{dy}{1+x^2+y^2} \right) dx = \int_B \frac{1}{1+x^2+y^2}\, dA$$

olur.

Bu bölge (ve $\frac{1}{1+x^2+y^2}$ fonksiyonu) kutupsal koordinatlarda işlem yapmaya daha uygundur. Kutupsal koordinatlarda, $B : \frac{\pi}{3} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2$ şeklindedir. İki katlı integrallerde değişken değişikliği teoreminden

$$\int_B \frac{1}{1+x^2+y^2}\, dA = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_0^2 \frac{1}{1+r^2}\, r\, dr \right) d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left. \frac{\ln(1+r^2)}{2} \right|_0^2 d\theta = \frac{\pi \ln 5}{12}$$

2.



İntegrasyon bölgesi (ikinci Tip olarak):
$B : 0 \le y \le 2, \quad y \le x \le \frac{y+2}{2}$ olarak yazılabilir.

$\bar{x} = \dfrac{\int_B x\rho(x,y)\, dA}{\int_B \rho(x,y)\, dA} \quad \bar{y} = \dfrac{\int_B y\rho(x,y)\, dA}{\int_B \rho(x,y)\, dA}$ dir.

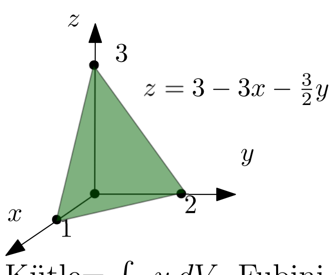
Fubini Teoreminden (her sürekli f fonksiyonu için):

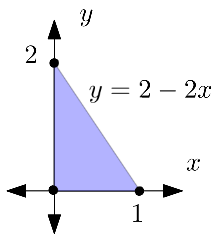
$$\int_B f(x,y)\, dA = \int_0^2 \left(\int_y^{\frac{y+2}{2}} f(x,y)\, dx \right) dy \text{ dir.}$$

$$\int_B \rho(x,y)\, dA = \int_0^2 \left(\int_y^{\frac{y+2}{2}} y\, dx \right) dy = \int_0^2 \left(y - \frac{y^2}{2} \right) dy = \frac{2}{3}$$

$$\int_B y\rho(x,y)\, dA = \int_0^2 \left(\int_y^{\frac{y+2}{2}} y^2\, dx \right) dy = \int_0^2 \left(y^2 - \frac{y^3}{2} \right) dy = \frac{2}{3} \quad \bar{y} = 1$$

3.





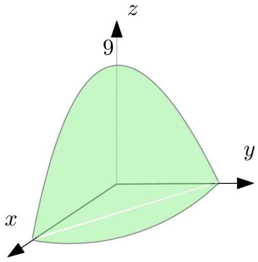
$(1,0,0),\ (0,2,0),\ (0,0,3)$ noktalarından geçen düzlemin denklemi $6x + 3y + 2z = 6$ dır. Buradan
$B = \{(x,y,z) : (x,y) \in B',\ 0 \le z \le 3 - 3x - \frac{3}{2}y\}$
(I. Tip olarak) $B' : 0 \le x \le 1 \quad 0 \le y \le 2 - 2x$
(B' aynı zamanda II. Tip bölgedir.)

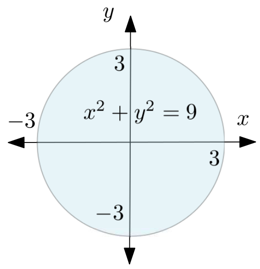
Kütle= $\int_B y\, dV$. Fubini nin Teoreminden,

$$\text{Kütle} = \int_{B'} \left(\int_0^{3-3x-\frac{3}{2}y} y\, dz \right) y\, dA = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} y\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} \left(3y - 3xy - \frac{3}{2}y^2 \right) dy\, dx = \int_0^1 \left(\frac{3}{2}(2-2x)^2 - \frac{3}{2}x(2-2x)^2 - \frac{1}{2}(2-2x)^3 \right) dx = \frac{1}{2}$$

4.





Kütle$= \int_B \mu(x,y,z)\,dV = \int_B \sqrt{x^2+y^2}\,dV$

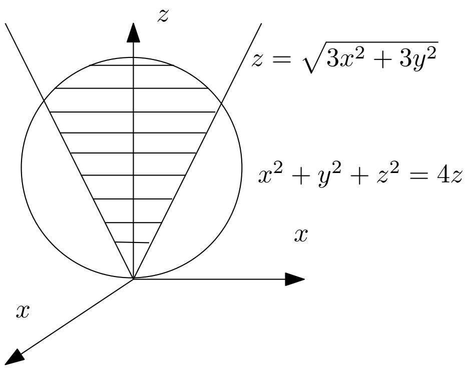
$B = \{(x,y,z) : (x,y) \in B',\ 0 \le z \le 9 - x^2 - y^2\}$
$B' = \{(x,y) : -3 \le x \le 3,\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}$
(Silindirik Koordinatlarda)

$B = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 3,\ 0 \le z \le 9 - r^2\}$

$$\int_B \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^3 \left(\int_0^{9-r^2} r\,dz\right) r\ dr\,d\theta = \int_0^{2\pi}\int_0^3 (9r^2 - r^4)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(3r^3 - \frac{r^5}{5}\right)\bigg|_0^3 d\theta = \frac{324\pi}{5}$$

5.



Küresel koordinatlarda:
Küre: $\rho^2 = 4\rho\cos\phi, \quad \rho = 4\cos\phi$
Koni: $\rho\cos\phi = \sqrt{3}\,\rho\sin\phi, \quad \tan\phi = \frac{1}{\sqrt{3}}, \quad \phi = \frac{\pi}{6}$

$$\text{Kütle} = \int_B \mu(x,y,z)\,dV = \int_B \rho\,dV$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{4\cos\phi} \rho\cdot\rho^2\sin\phi\ d\rho\,d\phi\,d\theta$$

$$\text{Kütle} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{4\cos\phi} \rho^3\sin\phi\ d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \frac{\rho^4}{4}\sin\phi\bigg|_0^{4\cos\phi} d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} 64\cos^4\phi\sin\phi\ d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \left(\frac{-64}{5}\cos^5\phi\right)\bigg|_0^{\frac{\pi}{6}} d\theta = \frac{128\pi}{5}\left(1 - \frac{9\sqrt{3}}{32}\right)$$